Model wavefunctions for Chiral Topological Order Interfaces

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Topological phases of matter [TOPMAT]: from the quantum Hall effect to spin liquids
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V. Crépel et al., arXiv:1806.06858
V. Crépel et al., PRB 97, 165136 (2018)
Motivations

What’s going on at the interface between two topologically ordered phases?

Topological order 1  ?  Topological order 2
Motivations

What’s going on at the interface between two topologically ordered phases?

Topological order 1 ? Topological order 2

Non-Abelian Anyons: When Ising Meets Fibonacci

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We consider an interface between two non-Abelian quantum Hall states: the Moore-Read state, supporting Ising anyons, and the $k = 2$ non-Abelian spin-singlet state, supporting Fibonacci anyons. It is shown that the interface supports neutral excitations described by a $(1+1)$-dimensional conformal field theory with a central charge $c = 7/10$. We discuss effects of the mismatch of the quantum statistical properties of the quasiholes between the two sides, as reflected by the interface theory.
Motivations

What’s going on at the interface between two topologically ordered phases?

Two questions we want to address:

- Can you build accurate model wavefunctions for the full system (bulk+interface)?
- Can we characterize the interface down to the microscopic level?
Outline

- Fractional Quantum Hall (FQH) model wavefunctions
- Matrix Product States (MPS) for the FQH model wavefunctions
- Building a model state for the Laughlin/Halperin interface
- Characterizing the interface
FQH (Abelian) model states
A single electron in 2D and in a $\perp$ magnetic field $B$.

**Uniform $\perp$ magnetic field**: gauge choice

$$H = \frac{1}{2m} \left( \vec{p} - e\vec{A} \right)^2, \quad \vec{A} = \frac{B}{2} \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$H = \frac{1}{2m} \left( -i\hbar \frac{\partial}{\partial x} + \frac{eB}{2} y \right)^2 + \frac{1}{2m} \left( -i\hbar \frac{\partial}{\partial y} - \frac{eB}{2} x \right)^2$$

- **energy scale**: cyclotron frequency $\omega_c = \frac{|eB|}{m}$,
- **length scale**: magnetic length $l_B = \frac{\hbar}{|eB|}$

$$H = \frac{1}{2} \hbar \omega_c \left[ \left( -i l_B \frac{\partial}{\partial x} + \frac{y}{2l_B} \right)^2 + \left( -i l_B \frac{\partial}{\partial y} - \frac{x}{2l_B} \right)^2 \right]$$
Landau levels

In (dimensionless) complex coordinate $z = (x + iy)/l_B$, and setting

$$a = \sqrt{2} \left( \frac{\partial}{\partial \bar{z}} + \frac{z}{2} \right), \quad a^\dagger = -\sqrt{2} \left( \frac{\partial}{\partial z} - \frac{\bar{z}}{2} \right)$$

Familiar form of the Hamiltonian

$$H = \hbar \omega_c \left( a^\dagger a + \frac{1}{2} \right) \quad [a, a^\dagger] = 1$$

$(n + 1)^{th}$ Landau level:

$$E_n = \hbar \omega_c \left( n + \frac{1}{2} \right)$$

Discrete spectrum, large degeneracy (translation invariance/guiding center).
Cylinder with perimeter $L$ (we identify $y \equiv y + L$)

Natural gauge choice: $\vec{A} = B \begin{pmatrix} 0 \\ x \end{pmatrix}$

$$t_y |\psi_{k_y}\rangle = k_y |\psi_{k_y}\rangle,$$

$$k_y = \frac{2\pi n}{L}$$

$$LLL$$

$$\Psi_{k_y}(x, y) = e^{iyk_y} e^{-\frac{(x - l_B^2 k_y)^2}{2l_B^2}}$$

Momentum $k_y$ and position $x$ are locked:

$$x \sim l_B^2 k_y$$

- $[\hat{x}, \hat{y}] = il_B^2$ implies that $\hbar \hat{x} = l_B^2 \hat{p}_y$.
- localized in $\hat{x}$ and delocalized in $\hat{y}$
- the interorbital distance is $\frac{2\pi}{L} l_B^2$

Density profile of the LLL orbital $\Psi_{k_y}(x, y)$. 

[$\hat{x}, \hat{y}$] = $il_B^2$ implies that $\hbar \hat{x} = l_B^2 \hat{p}_y$. 

localized in $\hat{x}$ and delocalized in $\hat{y}$

the interorbital distance is $\frac{2\pi}{L} l_B^2$
Projection to the LLL: dimensional reduction

Projection to the LLL: \(x\) and \(y\) no longer commute \([\hat{x}, \hat{y}] = i l_B^2\).

**4 dimensional phase space \(\Rightarrow 2\) dimensional phase space**

A **basis** of LLL states looks like a one-dimensional chain

But!

Physical short range interactions become long range in this description (distance of order \(l_B\) means \(\sim L/l_B\) sites).
Fractional Quantum Hall effect

Landau levels (spinless case)

Partial filling + interaction $\rightarrow$ FQHE
Filling factor: $\nu = \frac{hn}{eB} = \frac{N}{N_\Phi}$

$N$-body wave function:

$$\Psi = P(z_1, \ldots, z_N)e^{-\sum |z_i|^2/(4l_B^2)}$$

where $P$ is a polynomial.

Equivalently in occupation basis

$$|\psi\rangle = \sum_{\{m_i\}} c_{\{m_i\}} |m_1, \ldots, m_{N_{\text{orb}}}\rangle$$

How to guess $|\psi\rangle$? ED, DMRG, model wavefunctions, ....
The mother of all model wavefunctions

The $\nu = 1/3$ Laughlin state.

**filling fraction $\nu = 1/3$ + short range model interaction**

$\Rightarrow$ exact ground-state:

$$\Psi_{\frac{1}{3}}(z_1, \cdots, z_N) = \prod_{i<j}(z_i - z_j)^3 e^{-\sum_i |z_i|^2/4l_B^2}$$

The model interaction is the short range part of Coulomb.

**Extremely high overlap with Coulomb interaction!**

(Obtained by exact diagonalization)

First hints of a topological phase:

- excitations with fractional charge $e/3$
- topology dependent ground state degeneracy: $3^g$ exact ground states.
Very small cylinder perimeter $L$ : **LLL orbitals no longer overlap**

1d problem

Laughlin’s Hamiltonian $\rightarrow$ Haldane’s exclusion statistics

no more than 1 particle in three orbitals

At filling fraction $\nu = 1/3$, we get three possible states

\[
|\Psi_1\rangle = |\cdots 100100100\cdots\rangle \\
|\Psi_2\rangle = |\cdots 010010010\cdots\rangle \\
|\Psi_3\rangle = |\cdots 001001001\cdots\rangle
\]

3-fold degenerate ground state on the cylinder (and torus).
Metallic boundary: massless edge modes

\[ \psi_u = P_u(z_1, \cdots, z_N) \prod_{i<j} (z_i - z_j)^3 \]

where \( P_u \) is any symmetric, homogeneous polynomial.

Cartoon picture: no more than 1 electron in 3 orbitals.

- dispersion relation:
  \[ E = \nu_F \Delta P = \nu_F \frac{2\pi}{L} \Delta N \]
  chiral and gapless edge

- Number of edge states:
  - \( E = 0 \): 1 state
  - \( E = 1 \): 1 state
  - \( E = 2 \): 2 states
  - \( E = 3 \): 3 states
  - \( E = 4 \): 5 states
  - \( E = 5 \): 7 states
  - \( \cdots \)

  spectrum of a compact chiral boson \( \left( R = \sqrt{3} \right) \).
Metallic boundary: massless edge modes

\[ \psi_u = P_u \prod_{i<j} (z_i - z_j)^3 \]

where \( P_u \) is any symmetric, homogeneous polynomial.

Cartoon picture: no more than 1 electron in 3 orbitals.

- dispersion relation:
  \[ E = v_F \Delta P = v_F \frac{2\pi}{L} \Delta N \]
  chiral and gapless edge

- Number of edge states:

  spectrum of a compact chiral boson \((R = \sqrt{3})\).
Model states and CFT

- **Moore and Read**: A large set of model wavefunctions can be written as a CFT correlator (Laughlin, Moore-Read, Read-Rezayi, Halperin...).

\[
\Psi(z_1, \cdots, z_N) = \langle O_{bg} V(z_1) \cdots V(z_N) \rangle
\]

with \( V(z) \) an operator/field in a chiral 1 + 1 CFT and \( O_{bg} \) is the background charge.

- **Bulk-edge correspondence**: The CFT used to describe the (gapped) bulk is identical to the CFT that describes the (gapless) edge.

- **Laughlin state**:
  - \( V(z) = \exp(i \sqrt{m} \Phi(z)) \), where \( \Phi(z) \) is a (compact) chiral boson
  - \( \langle \Phi(z_1) \Phi(z_2) \rangle = -\log(z_1 - z_2) \)
  - \( \langle V(z_1) \cdots V(z_N) \rangle = \prod_{i<j}(z_i - z_j)^m \)

- **Halperin state**: A two-component (compact chiral) boson.
MPS for the FQH model states
Entanglement entropy

Cut the system in two parts $A$ and $B$ (the boundary has length $L$).

The entanglement entropy is

$$S_A = -\text{Tr}(\rho_A \log \rho_A)$$

with $\rho_A$ the reduced density matrix.

For a topological phase:

$$S_A \sim \alpha L - \gamma, \quad \gamma = \log \mathcal{D}$$

where $\mathcal{D}$ is the quantum dimension.

For $\nu = 1/3$ Laughlin: $\mathcal{D} = \sqrt{3}$
Entanglement spectrum

Schmidt decomposition

$$|\psi\rangle = \sum_{\alpha} \exp(-\xi_{\alpha}/2) |A, \alpha\rangle \otimes |B, \alpha\rangle$$

$$\rho_A = \sum_{\alpha} \exp(-\xi_{\alpha}) |A, \alpha\rangle \langle A, \alpha|$$

Entanglement spectrum

Li and Haldane (2008):

spectrum of $$\xi = -\log \rho_A$$

(plot $$\xi$$ vs momentum)

⇒ Reproduces the physical edge spectrum!
Matrix Product States

Any state can be written as

$$|\Psi\rangle = \sum_{\{m_i\}} \langle\alpha_L| A^{[m_1]}...A^{[m_{N_{\text{orb}}}]arial} |\alpha_R\rangle |m_1, \ldots, m_{N_{\text{orb}}}\rangle$$

- \(\{A^{[m]}\}\) is a set of \(\chi \times \chi\) matrices
- \((\alpha_l, \alpha_r)\) encode the boundary conditions for an open system.

The \(A^{[m]}_{\alpha,\beta}\) matrices have two types of indices

- \([m]\) is the physical index \((m \in \{0, 1\} \text{ for fermions, } m \in \mathbb{N} \text{ for bosons, } m \in \{\uparrow, \downarrow\} \text{ for spins ...})\)
- \((\alpha, \beta)\) are the bond indices (auxiliary space), ranging from \(1, \ldots, \chi\).
- The **bond dimension** \(\chi\) is of the order of \(\exp S_A\)
  - \(\Rightarrow\) for 2d gapped phases, it grows exponentially with \(L\).
  - An exponential improvement over the \(\exp(\text{surface})\) of ED...
Starting from a model wavefunction given by a CFT correlator

\[ \Psi(z_1, \cdots, z_N) = \langle u | O_{b.c.} V(z_1) \cdots V(z_N) | v \rangle \]

and expanding \( V(z) = \sum_n V_{-n} z^n \), one finds (up to orbital normalization)

\[ c(m_1, \cdots, m_n) = \langle u | O_{b.c.} \frac{1}{\sqrt{m_n !}} V_{-n}^{m_n} \cdots \frac{1}{\sqrt{m_2 !}} V_{-2}^{m_2} \frac{1}{\sqrt{m_1 !}} V_{-1}^{m_1} | v \rangle \]

This is a site/orbital dependent MPS

\[ c(m_1, \cdots, m_n) = \langle u | O_{b.c.} B[m_n](n) \cdots B[m_2](2) B[m_1](1) | v \rangle \]

with matrices at site/orbital \( j \) (including orbital normalization)

\[ B[m](j) = e^{(\frac{2\pi}{L} j)^2} \frac{(V_{-j})^m}{\sqrt{m !}} \]
A relation of the form $B[m](j) = U^{-1} B[m](j - 1) U$ yields

$$B[m](j) = U^{-j} B[m](0) U^j$$

and then

$$B[m_n](n) \cdots B[m_1](1) = U^{-n} \times B[m_n](0) U \cdots B[m_1](0) U$$

This is a translation invariant MPS, with matrices

$$A[m] = B[m](0) U$$
Translation invariant MPS on the cylinder

Site independent MPS

\[ B[m](j) = \frac{e^{\left(\frac{2\pi}{L}j\right)^2}}{\sqrt{m!}} (V-j)^m \quad \Rightarrow \quad A[m] = \frac{1}{\sqrt{m!}} (V_0)^m U \]

where \( U \) is the operator is (Zaletel and Mong (2012))

\[ U = e^{-\frac{2\pi}{L}H-i\sqrt{\nu}\varphi_0} \]

where

- \( \varphi_0 \) is the bosonic zero mode (\( e^{-i\sqrt{\nu}\varphi_0} \) shifts the electric charge by \( \nu \))
- \( H \) is the CFT cylinder Hamiltonian : \( H = \frac{2\pi}{L} L_0 \)
- \( V_0 \) is the zero mode of \( V(z) \)

auxiliary space = CFT Hilbert space
infinite bond dimension :/

Extension to spinfull FQH : V. Crépel et al., PRB 97, 165136 (2018)
Truncation of the auxiliary CFT basis

$\chi$ is infinite $\rightarrow$ a truncation scheme is required.

- The natural cut-off is the total conformal dimension $\rightarrow P_{\text{max}}$.
- Truncation over the momentum in the OES.
- In finite size, the truncated MPS becomes exact for $P_{\text{max}}$ large enough.

- DMRG: cut-off in $\xi$ (remove the smallest weight of $\rho_A$).
- MPS: cut-off in momentum.

Morally equivalent as long as the ES mimics the chiral edge mode spectrum (linear dispersion).
Building a model state for the Halperin/Laughlin interface
Let’s look at the following interface

We consider the (bosonic) case interfacing the Halperin (221) and the Laughlin $\nu = 1/2$ phases.

**Halperin (221)**
- Spinful, SU(2) symmetric, $\nu_\uparrow = \nu_\downarrow = 1/3$.
- $e/3$ excitations.
- Two $U(1)$ chiral edge modes (charge and spin).

**Laughlin $\nu = 1/2$**
- Spinless, $\nu_\uparrow = 0, \nu_\downarrow = 1/2$.
- $e/2$ excitations.
- One $U(1)$ chiral edge mode (charge).
A microscopic model

\[ H_{\text{int}} = \int d^2 \vec{r} \left( \sum_{\sigma, \sigma' = \uparrow, \downarrow} : \rho_\sigma(\vec{r}) \rho_{\sigma'}(\vec{r}) : \right) + \mu_\uparrow(\vec{r}) \rho_\uparrow(\vec{r}) \]

- Use the chemical potential \( \mu_\uparrow(\vec{r}) \) to polarize half of the system.
- Laughlin \( \nu = 1/2 \) is the densest polarized zero energy state.
- Halperin (221) is the densest unpolarized zero energy state.
- The two quantum liquids are sewed together by the interaction.

What shall we observe at the interface? A single gapless mode described by a free chiral boson (Haldane, PRL 94).
MPS and variational Ansatz

- We know the exact MPS for Halperin $B[n]$ and Laughlin $A[n]$.

- Brutal gluing: $\langle \alpha_L | \cdots B^{[m-2]} B^{[m-1]} A^{[m_0]} A^{[m_1]} \cdots | \alpha_R \rangle$

- Does $B^{[m-1]} A^{[m_0]}$ make any sense?
- Yes: conformal embedding!
- A careful choice for the electron, bg charge operators.
Translation invariance along the cylinder perimeter.

We recover the spin up and down densities in the bulk both on the Halperin and Laughlin side.

Finite size effects (with respect to $L$) quickly vanish.

Width of interface $\sim 5\ell_B$
We extract the TEE $\gamma$ from the derivative $S_A - L \partial_L S_A = -\gamma$.

Good agreement with the predicted values deep in the bulks (-0.549 and -0.347).
Up to a small oscillations (finite size effects more important for subleading terms), a rather smooth transition between the two bulk TEE.

No sign of the gapless mode (as recently predicted by Santos et al. arXiv:1803.04418).
Area law at the transition

Does we still satisfy the area law at the interface?

Yes (but hard to spot any deviation with such a limited range).
Characterizing the interface
Extracting $c$: Levin-Wen cut

$$\alpha(x_1)\ell + \alpha(x_2)(L - \ell) + 2 \int_{x_1}^{x_2} \alpha(u)du + \frac{c}{6} \log \left[ \sin \left( \frac{\pi \ell}{L} \right) \right] + K(w)$$

- $K(w)$ contains corrections to the area law, corner contributions,...
- Using a Levin-Wen cut to focus on the critical contribution.

$$S_A(\ell, w) = 2\frac{c}{6} \log \left[ \sin \left( \frac{\pi \ell}{L} \right) \right] + f(w)$$

- To get rid of $f(w)$ (including the TEE), we compute $S_A(\ell, w) - S_A(L/2, w)$
Extracting $c$ : Levin-Wen cut

Fitted central charge $c = 0.987(1)$. 
What about the bulk?

**Halperin**

![Graph](image1)

**Laughlin**

![Graph](image2)

**Interface**

![Graph](image3)
Compactification radius, fractional charge

- Central charge is only part of the information.
- Mutual information $\rightarrow$ full partition function of the CFT but hard to evaluate.
- Compactification radius $\leftrightarrow$ charge of the elementary edge excitation.
- Directly measure the charge along the edge.
- Play with the MPS boundary conditions.

\[ \langle \alpha_L \rangle \quad \ldots \ldots \quad \frac{2\pi \ell_B^2}{L} \quad \ldots \ldots \quad |\alpha_R\rangle \]

Halperin edge excitations
Laughlin edge excitations + interface excitations

Halperin Matrices, Laughlin Matrices
Excitations with a $e/6$ charge ($e/6 = e/2 - e/3$).

Edge = compact boson with radius $R = \sqrt{6}$. 
Is it a good variational wavefunction?

- It has all the features that we expect but *does it capture the microscopic model low energy properties?*

- Overlap with ED: $4 \uparrow + 9 \downarrow$ particles, 21 orbitals $\rightarrow 0.998$ (Hilbert space dim $\simeq 2.2 \times 10^8$).

ED with 15 orbitals.

**MPO** $L = 12$. 
Fermions: Laughlin $\nu = 1/3$ / Halperin (332)

- No conceptual difference with the bosonic example.
- Transition from Laughlin $\nu = 1/3$ to Halperin (332) at $\nu = 2/5$.
- Experimental relevance: graphene using the valley degree of freedom (spontaneous polarization at $\nu = 1/3$).
A variational ansatz to describe the interface between the Halperin and the Laughlin liquids.

Microscopic characterization of the interface gapless mode (c and R).

This scheme can be extended to

- any case where a MPS/PEPS/TN description is known on both sides.
- other sewing approach (e.g. superconductor).
- a more generic approach?