Model wavefunctions for Chiral Topological Order Interfaces

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Topological phases of matter [TOPMAT]: from the quantum Hall effect to spin liquids

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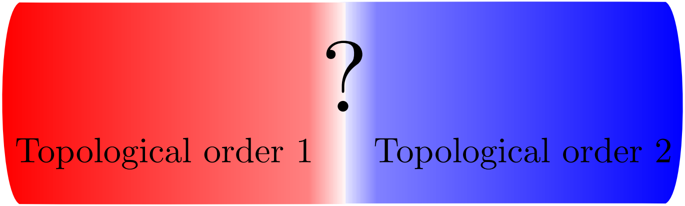
V. Crépel et al., arXiv :1806.06858
V. Crépel et al., PRB 97, 165136 (2018)

Motivations

What's going on at the interface between two topologically ordered phases?



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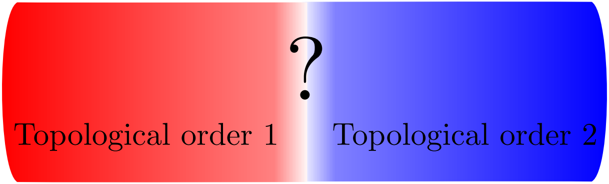

PRL **103,** 076803 (2009) PHYSICAL REVIEW LETTERS week ending 14 AUGUST 2009

Non-Abelian Anyons: When Ising Meets Fibonacci

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We consider an interface between two non-Abelian quantum Hall states: the Moore-Read state, supporting Ising anyons, and the $k = 2$ non-Abelian spin-singlet state, supporting Fibonacci anyons. It is shown that the interface supports neutral excitations described by a $(1 + 1)$-dimensional conformal field theory with a central charge $c = 7/10$. We discuss effects of the mismatch of the quantum statistical properties of the quasiholes between the two sides, as reflected by the interface theory.

Motivations

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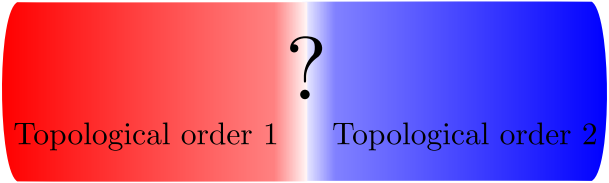


Two questions we want to address:

- Can you build accurate model wavefunctions for the full system (bulk+interface)?
- Can we characterize the interface down to the microscopic level?

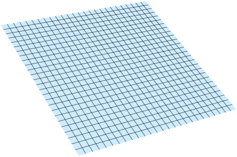
Outline

- Fractional Quantum Hall (FQH) model wavefunctions
- Matrix Product States (MPS) for the FQH model wavefunctions
- Building a model state for the Laughlin/Halperin interface
- Characterizing the interface

FQH (Abelian) model states

A single electron in 2D and in a $\perp$ magnetic field B.

Uniform $\perp$ magnetic field : gauge choice



$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2, \qquad \vec{A} = \frac{B}{2}\begin{pmatrix} -y \\ x \end{pmatrix}$$

$$H = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x} + \frac{eB}{2}y\right)^2 + \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial y} - \frac{eB}{2}x\right)^2$$

- **energy scale** cyclotron frequency $\omega_c = \frac{|eB|}{m}$,
- **length scale** : magnetic length $l_B = \sqrt{\frac{\hbar}{|eB|}}$

$$H = \frac{1}{2}\hbar\omega_c\left[\left(-il_B\frac{\partial}{\partial x} + \frac{y}{2l_B}\right)^2 + \left(-il_B\frac{\partial}{\partial y} - \frac{x}{2l_B}\right)^2\right]$$

Landau levels

In (dimensionless) complex coordinate $z = (x + iy)/l_B$, and setting

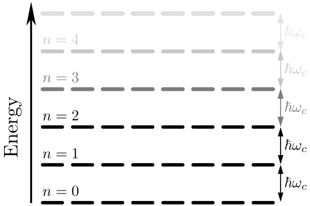
$$a = \sqrt{2}\left(\frac{\partial}{\partial \bar{z}} + \frac{z}{2}\right), \qquad a^\dagger = -\sqrt{2}\left(\frac{\partial}{\partial z} - \frac{\bar{z}}{2}\right)$$

Familiar form of the Hamiltonian

$$H = \hbar\omega_c\left(a^\dagger a + \frac{1}{2}\right) \qquad [a, a^\dagger] = 1$$

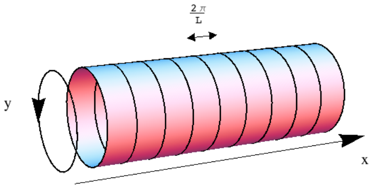
$(n+1)^{\text{th}}$ Landau level :

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right)$$



Discrete spectrum, large **degeneracy** (translation invariance/guiding center).

Cylinder with perimeter L (we identify $y \equiv y + L$)



Natural gauge choice : $\vec{A} = B \begin{pmatrix} 0 \\ x \end{pmatrix}$

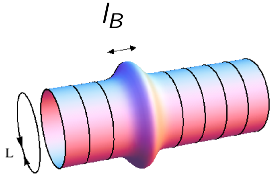
$$t_y|\Psi_{k_y}\rangle = k_y|\Psi_{k_y}\rangle, \qquad k_y = \frac{2\pi n}{L}$$

LLL $\qquad \Psi_{k_y}(x,y) = e^{iyk_y} e^{-\frac{(x-l_B^2 k_y)^2}{2l_B^2}}$

Momentum k_y and position x are locked :

$$x \sim l_B^2 k_y$$

- $[\hat{x}, \hat{y}] = il_B^2$ implies that $\hbar\hat{x} = l_B^2 \hat{p}_y$.
- localized in $\hat{x}$ and delocalized in $\hat{y}$
- the interorbital distance is $\frac{2\pi}{L} l_B^2$



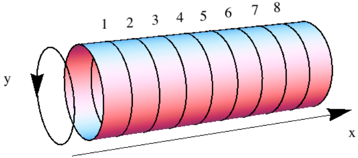
Density profile of the LLL orbital $\Psi_{k_y}(x,y)$.

Projection to the LLL : dimensional reduction

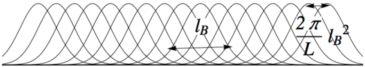
Projection to the LLL : x and y no longer commute $[\hat{x}, \hat{y}] = i\, l_B^2$.

4 dimensional phase space $\Rightarrow$ 2 dimensional phase space

A **basis** of LLL states



looks like a one-dimensional chain

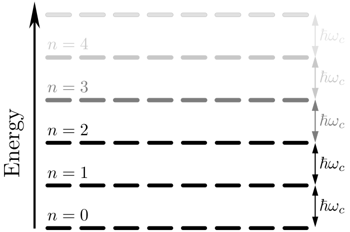


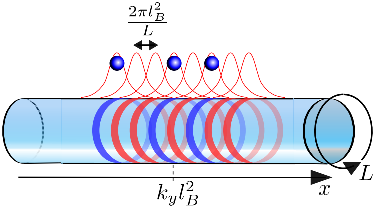
But !

Physical short range interactions become long range in this description (distance of order l_B means $\sim L/l_B$ sites).

Fractional Quantum Hall effect

Landau levels (spinless case)





- Partial filling + interaction → FQHE
- Filling factor : $\nu = \frac{hn}{eB} = \frac{N}{N_\Phi}$
- N-body wave function :

$$\Psi = P(z_1, ..., z_N) e^{-\sum |z_i|^2/(4l_B^2)}$$

  where P is a polynomial.
- Equivalently in occupation basis

$$|\Psi\rangle = \sum_{\{m_i\}} c_{\{m_i\}} \; |m_1, ..., m_{N_{\mathrm{orb}}}\rangle$$

- How to guess $|\Psi\rangle$? ED, DMRG, model wavefunctions, ...

The mother of all model wavefunctions

The $\nu = 1/3$ Laughlin state.

filling fraction $\nu = 1/3$ **+ short range model interaction**

$\Rightarrow$ exact ground-state :

$$\Psi_{\frac{1}{3}}(z_1, \cdots, z_N) = \prod_{i<j}(z_i - z_j)^3 \, e^{-\sum_i |z_i|^2/4l_B^2}$$

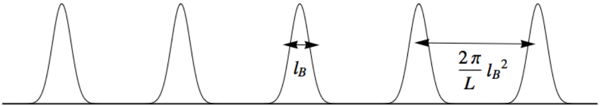
The model interaction is the short range part of Coulomb.

Extremely high overlap with Coulomb interaction !
(obtained by exact diagonalization)

First hints of a topological phase :

- excitations with fractional charge $e/3$
- topology dependent ground state degeneracy : 3^g exact ground states.

Cartoon picture : thin cylinder limit ($L \ll l_B$)



Very small cylinder perimeter L : **LLL orbitals no longer overlap**
1d problem

Laughlin's Hamiltonian $\rightarrow$ Haldane's exclusion statistics
no more than 1 particle in three orbitals

At filling fraction $\nu = 1/3$, we get three possible states

$$|\Psi_1\rangle = |\cdots 1\,0\,0\,1\,0\,0\,1\,0\,0 \cdots\rangle$$
$$|\Psi_2\rangle = |\cdots 0\,1\,0\,0\,1\,0\,0\,1\,0 \cdots\rangle$$
$$|\Psi_3\rangle = |\cdots 0\,0\,1\,0\,0\,1\,0\,0\,1 \cdots\rangle$$

3-fold degenerate ground state on the cylinder (and torus).

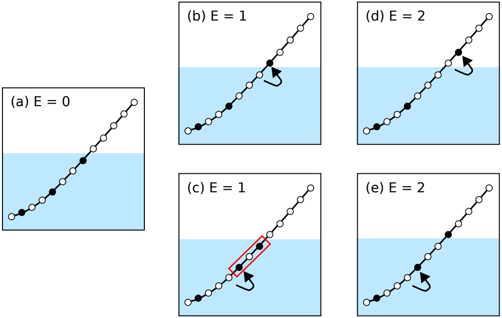
Metallic boundary : massless edge modes

$$\Psi_u = P_u(z_1, \cdots, z_N) \prod_{i<j} (z_i - z_j)^3$$

where P_u is any symmetric, homogeneous polynomial.

Cartoon picture: no more than 1 electron in 3 orbitals.

- dispersion relation :
  $E = v_F \Delta P = v_F \frac{2\pi}{L} \Delta N$
  chiral and **gapless** edge
- Number of edge states :
  - $E = 0$: 1 state
  - $E = 1$: 1 state
  - $E = 2$: 2 states
  - $E = 3$: 3 states
  - $E = 4$: 5 states
  - $E = 5$: 7 states
  - $\cdots$



(cartoon picture)

spectrum of a compact chiral boson ($R = \sqrt{3}$).

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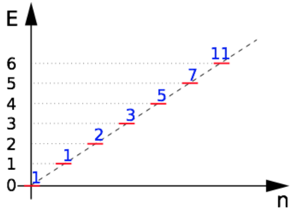
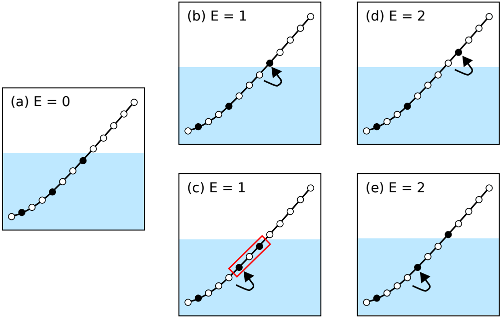




(cartoon picture)

spectrum of a compact chiral boson ($R = \sqrt{3}$).

Model states and CFT

- **Moore and Read :** A large set of model wavefunctions can be written as a CFT correlator (Laughlin, Moore-Read, Read-Rezayi, Halperin...).

$$\Psi(z_1, \cdots, z_N) \;=\; \langle \mathcal{O}_{\mathrm{bg}} V(z_1) \cdots V(z_N) \rangle$$

  with $V(z)$ an operator/field in a chiral $1+1$ CFT and $\mathcal{O}_{\mathrm{bg}}$ is the background charge.
- Bulk-edge correspondence : The CFT used to describe the (gapped) bulk is identical to the CFT that describes the (gapless) edge
- **Laughlin state** :
  - $V(z) =: \exp(i\sqrt{m}\Phi(z)) :$, where $\Phi(z)$ is a (compact) chiral boson
  - $\langle \Phi(z_1)\Phi(z_2) \rangle = -\log(z_1 - z_2)$
  - $\langle V(z_1) \cdots V(z_N) \rangle = \prod_{i<j}(z_i - z_j)^m$
- **Halperin state** : A two-component (compact chiral) boson.

MPS for the FQH model states

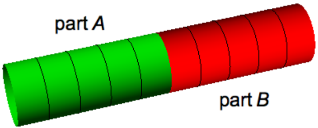
Entanglement entropy

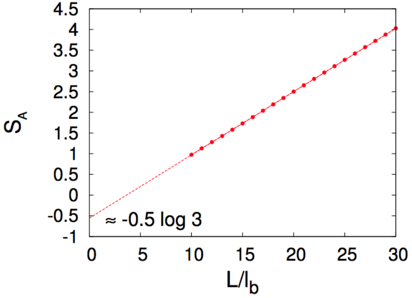
Cut the system in two parts A and B (the boundary has length L)

The **entanglement entropy** is

$$S_A = -\mathrm{Tr}(\rho_A \log \rho_A)$$

with ρ_A the reduced density matrix.





Entanglement entropy of the $\nu = 1/3$ Laughlin state as a function of the cylinder perimeter L (N. Regnault)

For a topological phase :

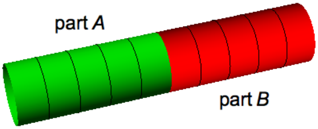
$$S_A \sim \alpha L - \gamma, \qquad \gamma = \log \mathcal{D}$$

where $\mathcal{D}$ is the quantum dimension.

For $\nu = 1/3$ Laughlin : $\mathcal{D} = \sqrt{3}$

Entanglement spectrum

Schmidt decomposition

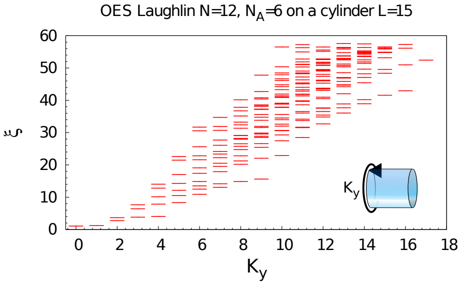


$$|\Psi\rangle = \sum_\alpha \exp(-\xi_\alpha/2)\,|{\color{green}A},\alpha\rangle \otimes |{\color{red}B},\alpha\rangle$$

$$\rho_A = \sum_\alpha \exp(-\xi_\alpha)\,|A,\alpha\rangle\,\langle A,\alpha|$$

Entanglement spectrum

Li and Haldane (2008):
spectrum of $\xi = -\log \rho_A$
(plot ξ vs momentum)



$\Rightarrow$ **Reproduces the physical edge spectrum !**

Matrix Product States

Any state can be written as

$$|\Psi\rangle = \sum_{\{m_i\}} \langle \alpha_L| A^{[m_1]} ... A^{[m_{N_{\rm orb}}]} |\alpha_R\rangle \; |m_1, ..., m_{N_{\rm orb}}\rangle$$

- $\{A^{[m]}\}$ is a set of $\chi \times \chi$ matrices
- (α_l, α_r) encode the boundary conditions for an open system.

The $A^{[m]}_{\alpha,\beta}$ matrices have two types of indices

- $[m]$ is the physical index ($m \in \{0, 1\}$ for fermions, $m \in \mathbb{N}$ for bosons, $m \in \{\uparrow, \downarrow\}$ for spins ...)
- (α, β) are the bond indices (auxiliary space), ranging from $1, ..., \chi$.
- The **bond dimension** χ is of the order of $\exp S_A$
  $\Rightarrow$ for 2d gapped phases, it grows exponentially with L.
  An exponential improvement over the $\exp(\mathrm{surface})$ of ED...

Starting from a model wavefunction given by a CFT correlator

$$\Psi(z_1, \cdots, z_N) = \langle u|\mathcal{O}_{\mathrm{b.c.}} V(z_1) \cdots V(z_N)|v\rangle$$

and expanding $V(z) = \sum_n V_{-n} z^n$, one finds (up to orbital normalization)

$$c_{(m_1,\cdots,m_n)} = \langle u|\, \mathcal{O}_{\mathrm{b.c.}} \frac{1}{\sqrt{m_n!}} V_{-n}^{m_n} \cdots \frac{1}{\sqrt{m_2!}} V_{-2}^{m_2} \frac{1}{\sqrt{m_1!}} V_{-1}^{m_1} \,|v\rangle$$

This is a site/orbital dependent MPS

$$c_{(m_1,\cdots,m_n)} = \langle u|\, \mathcal{O}_{\mathrm{b.c.}} B^{[m_n]}(n) \cdots B^{[m_2]}(2) B^{[m_1]}(1) \,|v\rangle$$

with matrices at site/orbital j (including orbital normalization)

$$\boxed{B^{[m]}(j) = \frac{e^{\left(\frac{2\pi}{L} j\right)^2}}{\sqrt{m!}} \left(V_{-j}\right)^m}$$

Translation invariant MPS

A relation of the form $B^{[m]}(j) = U^{-1} B^{[m]}(j-1) U$ yields

$$B^{[m]}(j) = U^{-j} B^{[m]}(0) U^{j}$$

and then

$$B^{[m_n]}(n) \cdots B^{[m_1]}(1) = U^{-n} \times B^{[m_n]}(0) U \cdots B^{[m_1]}(0) U$$

This is a **translation invariant MPS**, with matrices

$$A^{[m]} = B^{[m]}(0) U$$

Translation invariant MPS on the cylinder

Site independant MPS

$$B^{[m]}(j) = \frac{e^{\left(\frac{2\pi}{L}j\right)^2}}{\sqrt{m!}} \left(V_{-j}\right)^m \qquad \Rightarrow \qquad A^{[m]} = \frac{1}{\sqrt{m!}} \left(V_0\right)^m U$$

where U is the operator is (Zaletel and Mong (2012))

$$U = e^{-\frac{2\pi}{L}H - i\sqrt{\nu}\varphi_0}$$

where

- φ_0 is the bosonic zero mode ($e^{-i\sqrt{\nu}\varphi_0}$ shifts the electric charge by ν)
- H is the CFT cylinder Hamiltonian : $H = \frac{2\pi}{L} L_0$
- V_0 is the zero mode of $V(z)$

auxiliary space = CFT Hilbert space

infinite bond dimension :/

Extension to spinfull FQH : **V. Crépel et al., PRB 97, 165136 (2018)**

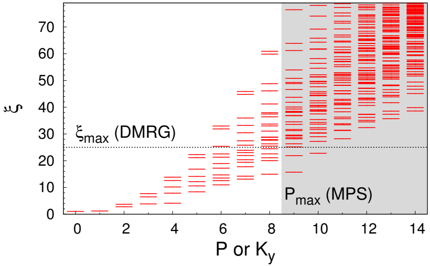
Truncation of the auxiliary CFT basis

χ is infinite $\rightarrow$ a truncation scheme is required.

- The natural cut-off is the total conformal dimension $\rightarrow P_{\max}$.
- Truncation over the momentum in the OES.
- In finite size, the truncated MPS becomes exact for $P_{\max}$ large enough.

- DMRG : cut-off in ξ (remove the smallest weight of ρ_A).
- MPS : cut-off in momentum.

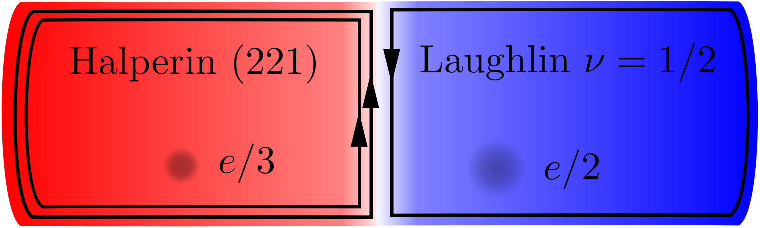
Morally equivalent as long as the ES mimics the chiral edge mode spectrum (linear dispersion).



Building a model state for the Halperin/Laughlin interface

Let's look at the following interface

We consider the (bosonic) case interfacing the Halperin (221) and the Laughlin $\nu = 1/2$ phases.



Halperin (221)

- Spinful, SU(2) symmetric, $\nu_\uparrow = \nu_\downarrow = 1/3$.
- $e/3$ excitations.
- Two $U(1)$ chiral edge modes (charge and spin).

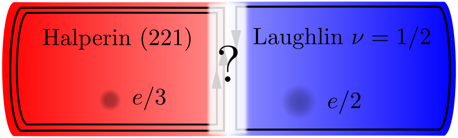
Laughlin $\nu = 1/2$

- Spinless, $\nu_\uparrow = 0, \nu_\downarrow = 1/2$.
- $e/2$ excitations.
- One $U(1)$ chiral edge mode (charge).

A microscopic model

$$\mathcal{H}_{\rm int} = \int \mathrm{d}^2\vec{r} \left(\sum_{\sigma,\sigma'=\uparrow,\downarrow} :\rho_\sigma(\vec{r})\rho_{\sigma'}(\vec{r}): \right) + \mu_\uparrow(\vec{r})\rho_\uparrow(\vec{r})$$

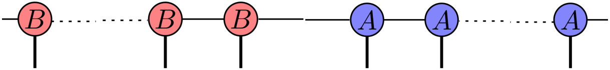
- Use the chemical potential $\mu_\uparrow(\vec{r})$ to polarize half of the system.
- Laughlin $\nu = 1/2$ is the densest polarized zero energy state.
- Halperin (221) is the densest unpolarized zero energy state.
- The two quantum liquids are sewed together by the interaction.



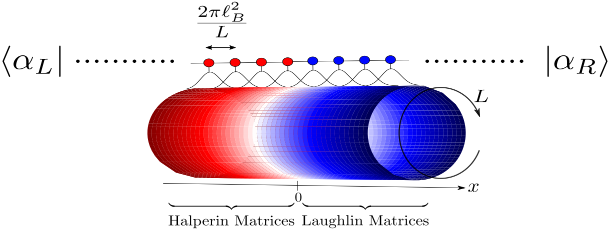
What shall we observe at the interface ? A single gapless mode described by a free chiral boson (Haldane, PRL 94).

MPS and variational Ansatz

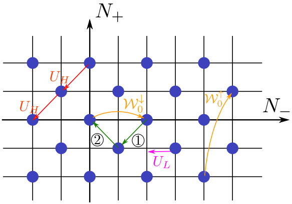
- We know the exact MPS for Halperin $B^{[n]}$ and Laughlin $A^{[n]}$.



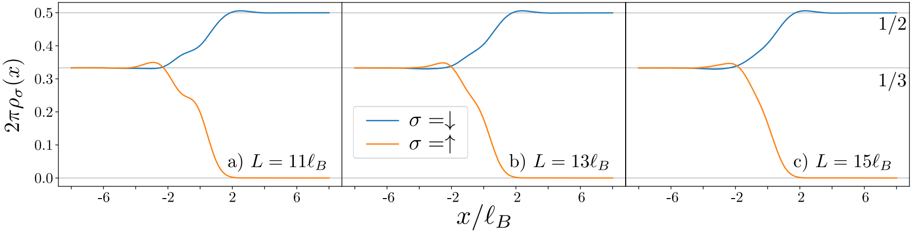
- Brutal gluing : $\langle \alpha_L | \cdots B^{[m_{-2}]} B^{[m_{-1}]} A^{[m_0]} A^{[m_1]} \cdots | \alpha_R \rangle$



- Does $B^{[m_{-1}]} A^{[m_0]}$ make any sense?
- Yes : **conformal embedding !**
- A careful choice for the electron, bg charge operators.

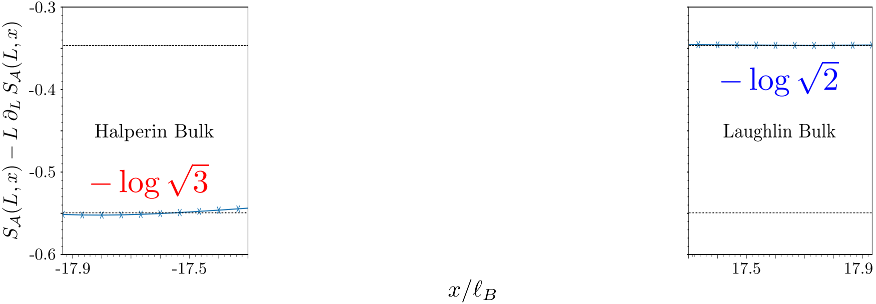


Density



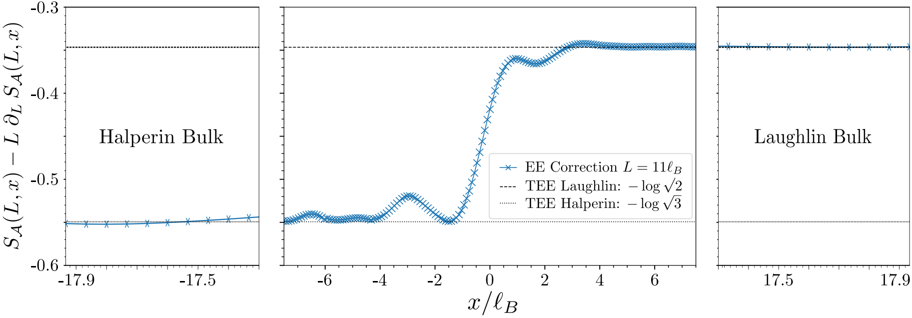
- Translation invariance along the cylinder perimeter.
- We recover the spin up and down densities in the bulk both on the Halperin and Laughlin side.
- Finite size effects (with respect to L) quickly vanish.
- Width of interface $\simeq 5 l_B$

Topological Entanglement Entropy



- We extract the TEE γ from the derivative $S_{\mathcal{A}} - L\partial_L S_{\mathcal{A}} = -\gamma$.
- Good agreement with the predicted values deep in the bulks (-0.549 and -0.347).

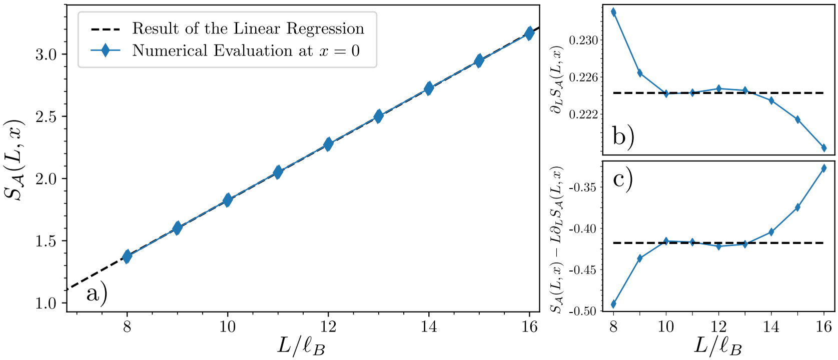
Topological Entanglement Entropy



- Up to a small oscillations (finite size effects more important for subleading terms), a rather smooth transition between the two bulk TEE.
- No sign of the gapless mode (as recently predicted by Santos et al. arXiv:1803.04418).

Area law at the transition

Does we still satisfy the area law at the interface?

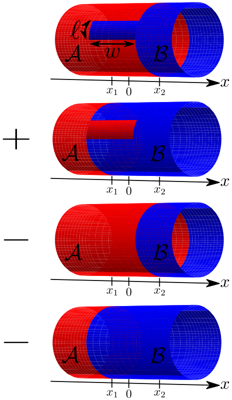


Yes (but hard to spot any deviation with such a limited range).

Characterizing the interface

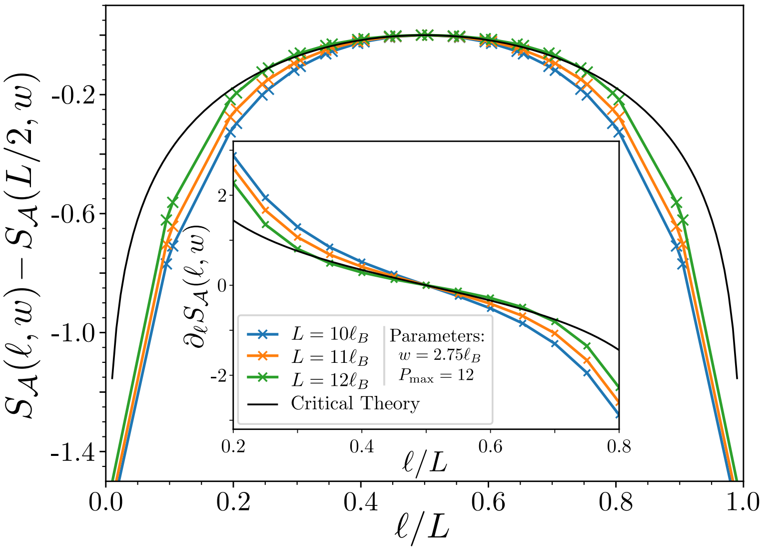
Extracting c : Levin-Wen cut

$$\underbrace{\alpha(x_1)\ell + \alpha(x_2)(L-\ell) + 2\int_{x_1}^{x_2} \alpha(u)\mathrm{d}u}_{\text{Area Law}} + \underbrace{\frac{c}{6}\log\left[\sin\left(\frac{\pi\ell}{L}\right)\right]}_{\text{Critical Mode}} + K(w)$$



- $K(w)$ contains corrections to the area law, corner contributions,...
- Using a Levin-Wen cut to focus on the critical contribution.
- $$S_{\mathcal{A}}(\ell, w) = 2\frac{c}{6}\log\left[\sin\left(\frac{\pi\ell}{L}\right)\right] + f(w)$$
- To get rid of $f(w)$ (including the TEE), we compute $S_{\mathcal{A}}(\ell, w) - S_{\mathcal{A}}(L/2, w)$

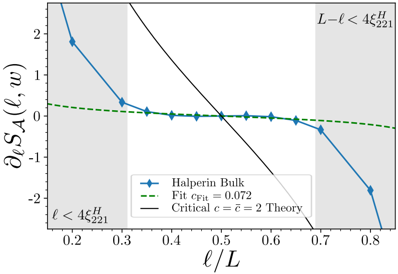
Extracting c : Levin-Wen cut



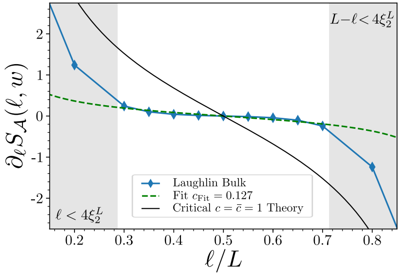
Fitted central charge $c = 0.987(1)$.

What about the bulk?

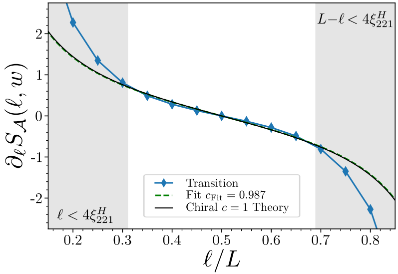
Halperin



Laughlin

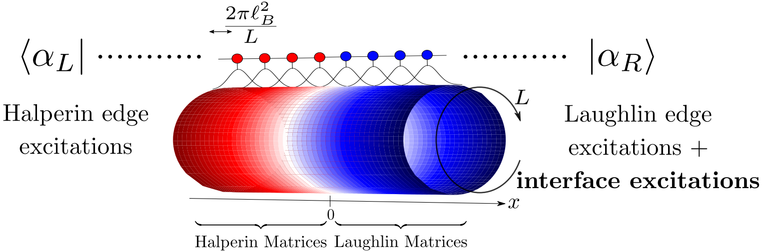


Interface

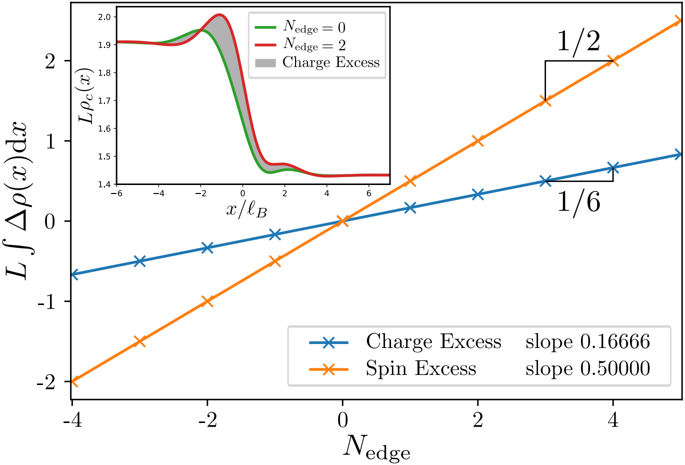


Compactification radius, fractional charge

- Central charge is only part of the information.
- Mutual information → full partition function of the CFT but hard to evaluate.
- Compactification radius ↔ charge of the elementary edge excitation.
- Directly measure the charge along the edge.
- Play with the MPS boundary conditions.



Compactification radius, fractional charge

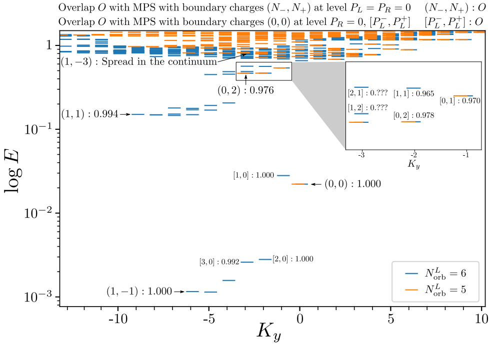


Excitations with a $e/6$ charge ($e/6 = e/2 - e/3$).

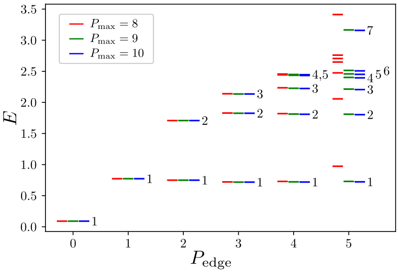
Edge = compact boson with radius $R = \sqrt{6}$.

Is it a good variational wavefunction?

- It has all the features that we expect but *does it capture the microscopic model low energy properties?*
- Overlap with ED : $4 \uparrow + 9 \downarrow$ particles, 21 orbitals $\rightarrow$ 0.998 (Hilbert space dim $\simeq 2.2 \times 10^8$).



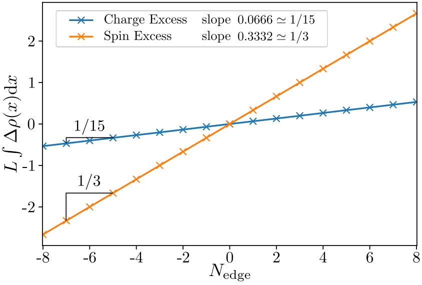
ED with 15 orbitals.

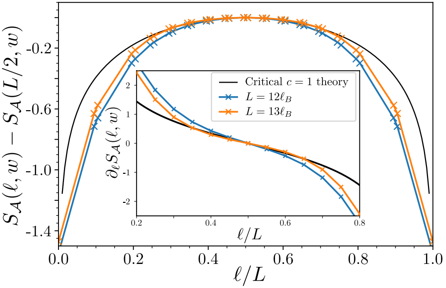


MPO $L = 12$.

Fermions : Laughlin $\nu = 1/3$ / Halperin (332)

- No conceptual difference with the bosonic example.
- Transition from Laughlin $\nu = 1/3$ to Halperin (332) at $\nu = 2/5$.
- Experimental relevance : graphene using the valley degree of freedom (spontaneous polarization at $\nu = 1/3$).





Conclusion

- A variational ansatz to describe the interface between the Halperin and the Laughlin liquids.
- Microscopic characterization of the interface gapless mode (c and R).
- This scheme can be extended to
  - any case where a MPS/PEPS/TN description is known on both sides.
  - other sewing approach (e.g. superconductor).
  - a more generic approach ?